Comparison of Judgment Scales of the Analytical Hierarchy Process - A New Approach

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 2017-07-06

Abstract

The analytic hierarchy process (AHP) remains a popular multi-criteria decision method. One topic under discussion of AHP is the use of different scales to translate judgments into ratios. The author makes a new approach to compare different scale functions and to derive a recommendation for the application of scales. The approach is based on simple analytic functions and takes into consideration the number of criteria of the decision problem. A correction to the so-called balanced scale is proposed, and a new adaptive-balanced scale introduced. Scales are then categorized and compared based on weight boundaries, weight ratio, weight uncertainties, weight dispersion and number of decision criteria. Finally a practical example of a decision hierarchy is presented applying the different scales. The results show that the corrected balanced scale improves weight dispersion and weight uncertainty in comparison to the original AHP scale. The proposed adaptive-balanced scale overcomes the problem of a change of the maximum weight depending on the number of decision criteria.

Keywords

Decision support systems, Multi-criteria decision making, Analytic Hierarchy Process, Pairwise Comparisons, Judgment scales.

1. Introduction

Despite all academic discussions, the analytic hierarchy process (AHP) remains one of the most popular multi-criteria decision making methods (MCDM). Originally proposed by Saaty (1980), over the last decades several modifications and improvement have been proposed. A review of the main developments in AHP can be found from Ishizaka & Labib (2011). One of the topics being under discussion for a long time is the fundamental AHP scale. Saaty and Vargas (2012) describe ratio scales, proportionality and normalized ratio scales as one of the seven pillars of the Analytic Hierarchy Process. The fundamental AHP scale of absolute numbers is derived from the psychophysical law of Weber–Fechner and uses absolute numbers 1, 2, 3, … 9 or its verbal equivalents (Table 1).

Judgment x	Verbal equivalent	Comment
1	*Equal importance*	Two activities contribute equally to the objective.
2	Weak or slight	
3	*Moderate importance*	Experience and judgment slightly favor one activity over another.
4	Moderate plus	
5	*Strong importance*	Experience and judgment strongly favor one activity over another.
6	Strong plus	
7	*Very strong or demo-strated importance*	An activity is favored very strongly over another; its dominance demonstrated in practice.
8	*Very, very strong*	
9	*Extreme importance*	The evidence favoring one activity over another is of the highest possible order of affirmation.

<u>Table 1</u>. Fundamental AHP judgment scale with integers 1 to 9 and their verbal equivalents.

Paired comparisons are made by identifying the less dominant of two elements and using it as the unit of measurement. One then determines, how many times more the dominant member of the pair is than this unit. The reciprocal value is used for the comparison of the less dominant element with the more dominant one.

Theoretically there is no reason to be restricted to these numbers and verbal gradation. Several other numerical scales, summarized in table 2 based on Ishizaka & Labib (2011), have been proposed, but there are no guidelines, what scale to use for a specific decision problem, and the choice of the "best" scale is an ongoing discussion. Franeka and Kresta (2014) classified the judgment scales based on consistency and allocation of priorities for a specific example with seven criteria, using the row geometric mean method (RGGM), instead of the eigenvalue (EV) method for the calculation of priorities.

In this paper the author makes a new approach. We look at the case of one extreme important criterion ($x = 9$ on the fundamental AHP judgment scale) compared to all others as a function of the number of criteria. For this specific case simple relations can be derived to compare the different scale functions. We have categorized the scales into three categories:

* *Category 1* – The maximum entry value in the decision matrix is kept at nine: linear AHP scale, inverse linear scale, balanced scale, balanced-n scale
* *Category 2* - The maximum range of entry values in the decision matrix is reduced to lower values than nine: logarithmic scale, root square scale, Koczkodaj scale.
* *Category 3* –The maximum entry values and range of entry values in the decision matrix is extended to higher values exceeding nine: power scale, geometric scale, adaptive and adaptive-balanced scale.

Cat		Name	Short	Scale function	M	Comment
1	1	Linear 1 to 9 AHP scale	AHP	$c = x$	9	Saaty (1980)
	2	Inverse linear scale	Inv-lin	$c = \dfrac{9}{10 - x}$	9	Ma-Zheng (1991)
	3	Balanced scale	Bal	$c = \dfrac{0.45 + 0.05x}{1 - 0.45 + 0.05x}$	9	Saalo, Hämäläinen (1997) for [01,0.9]
	4	Balanced-n scale*	Bal-n	See text, eq. 4	9	Corrected scale no. 3
2	5	Logarithmic Scale	Log	$c = \log_a(x + a - 1)$	3.3	Ishizaka et. al (2010) we use a=2
	6	Root square scale	Root	$c = \sqrt[a]{x}$	3	Harker, Vargas (1987) we use a=2
	7	Koczkodaj scale	Kocz	$c = 1 + \dfrac{x - 1}{9 - 1}$	2	Koczkodaj (2016)
3	8	Power scale	Power	$c = x^a$	81	Harker, Vargas (1987) we use a=2
	9	Geometric scale	Geom	$c = a^{x-1}$	256	Lootsma (1994), we use a=2
	10	Adaptive scale*	Adapt	See text, eq. 6	81	M for n=10 criteria
	11	Adaptive-bal scale*	Adapt-bal	See text, eq. 7	81	M for n = 10 criteria

Table 2. AHP scales investigated in this paper. x is the value on the integer judging scale for pairwise comparisons from 1 to 9, c the ratio used as entry into the decision matrix, M the maximum value of c for x = 9. Scales with * are explained in this paper.

The author focuses on intangible judgments on the fundamental scale (table 1), translated into ratios $1/M$ … 1 … M. We do not consider actual measurements like distance, area or temperature, where a limited scale is not necessary, and AHP can be applied using actual measured ratios.

In the following, we first will have a closer look at the balanced scale (table 2, no. 3), and introduce some corrections to it in order to reflect the dependency on the number of criteria. Then we will investigate all scales looking at the following criteria:

1. *Weight bound and weight ratio*: What is the *maximum weight* for a judgment that one criterion is "9 - extreme more important" than all others, and how compares the total *ratio* of calculated weights for different scales?
2. *Weight uncertainty*: How much depend the weights on *small variations of the judgment*?
3. *Weight dispersion*: How are the weights distributed over the judgment range?

2. Is the Balanced Scale balanced?

Salo & Hämäläinen (1997) pointed out that the integers from 1 to 9 yield local weights, which are not equally dispersed. Based on this observation, they proposed a balanced scale, where local weights are evenly dispersed over the weight range [0.1, 0.9]. They state that for a given set of priority vectors w the corresponding ratios r can be computed from the inverse relationship

$$r = w/(1 - w). \tag{1a}$$

The priorities 0.1, 0.15, 0.2, ... 0.8, 0.9 lead, for example, to the scale 1, 1.22, 1.5, 1.86, 2.33, 3.00, 4.00, 5.67 and 9.00. This scale can be computed by

$$w_{\text{bal}} = 0.45 + 0.05\,x \tag{1b}$$

with $x = 1 \ldots 9$ and

$$c = \frac{w_{\text{bal}}}{1-w_{\text{bal}}} = \frac{0.45+0.05x}{1-0.45+0.05x} \tag{1c}$$

c (resp. $1/c$) are the entry values in the decision matrix, and x the pairwise comparison judgment on the scale 1 to 9.

In fact, eq. 1a or its inverse are the *special case for one selected pairwise comparison* of two criteria. If we take into account the complete n x n decision matrix for n criteria, the resulting weights for one criterion, judged as x-times more important than all others, can be calculated as (see annex 1):

$$w_{\text{AHP}} = \frac{x}{x+n-1} \tag{2}$$

Eq. 2 simplifies to eq. 1a for n=2.
With eq. 2 we can formulate the general case for the balanced scale, resulting in evenly dispersed weights for n criteria and a judgment x with x from 1 to M:

$$w_{\text{bal}}(x) = w_{\text{eq}} + \frac{w_{\text{max}}-w_{\text{eq}}}{M-1}(x - 1) \tag{3}$$

with

$$w_{\text{eq}} = \frac{1}{n} \tag{3a}$$

$$w_{\text{max}} = \frac{M}{n+M-1} \tag{3b}$$

$$w_{\text{min}} = \frac{1}{n+M-1} \tag{3c}$$

We get the general balanced scale (we call it balanced-n scale) as

$$c = \frac{w_{\text{bal}}(x)}{1-w_{\text{bal}}(x)}(n - 1) \tag{4}$$

With n=2 and M=9 it represents the classical balanced scale as given in eq. 1b and 1c. Fig. 1 shows the weights as a function of judgements derived from a case with $n = 7$ criteria using the fundamental AHP, balanced and general balanced (bal-n) scale. It can be

seen that, for example, a single judgement "*5 – strong more important*" yields to a weight of 45% on the AHP scale, 28% on the balanced scale and 37% on the balanced-n scale.
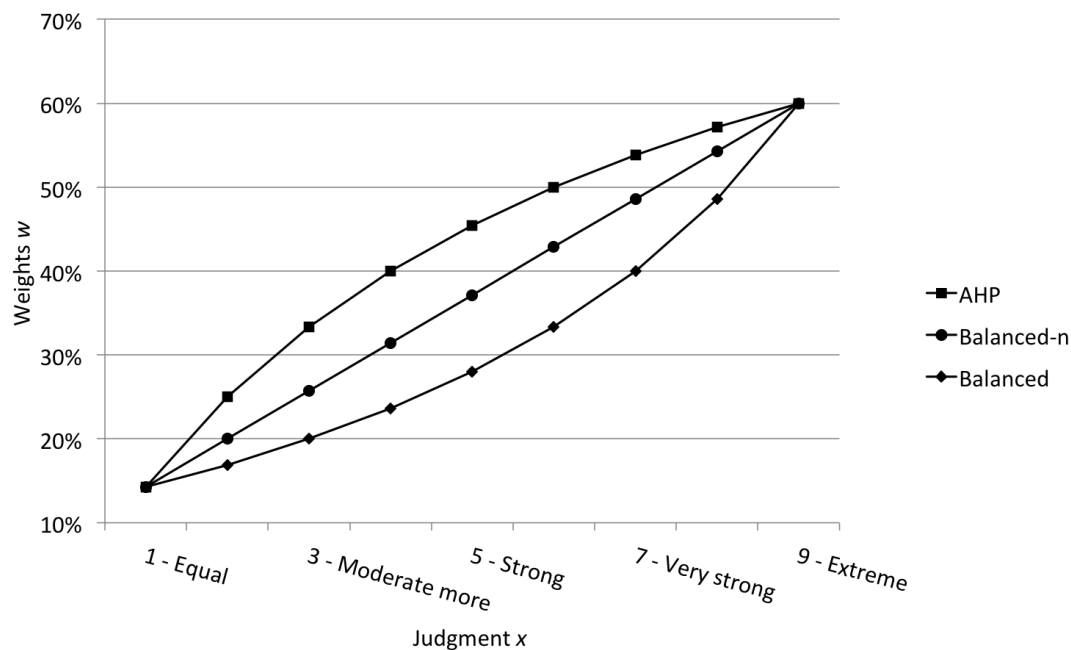

Figure 1. Weights as function of judgment for the AHP scale, the balanced scale and the corrected balanced scale for 7 decision criteria.

A "strong" criterion is underweighted using the classical balanced scale, and over weighted using the standard AHP scale, compared to the general balanced-n scale. Weights of the balanced-n scale are distributed evenly over the judgment range, and only for *n* = 2 the original proposed balanced scale is identical with the balanced-n scale and yields evenly distributed weights.

3. Weight boundaries and weight ratios

As already stated by Salo & Hämäläinen (1997) a shortcoming of any scale with a finite upper bound M is that the upper bound restricts the range of local priority vectors. The upper bound is given by eq. 3b. Scales in category 3 (table 2) extend the upper bound M from nine to higher values, scales in category 2 reduce the upper bound M from nine to lower values. For example, the highest priority for two criteria on the fundamental AHP scale is 90%, on the root square scale 75% and on the power scale 98.8%. The maximum weights w_{max} depend on the number of criteria (eq. 3b), and decrease with an increasing number of criteria. The resulting weights for the same judgement *9 - extreme more important* can vary from 90% (AHP scale, *n* = 2 criteria) to 52% (AHP scale, *n* = 9 criteria).

When judging a specific criterion as "extrem more important" (9 on the fundamental AHP scale) than all other criteria, a decision maker would expect the resulting weight for this specific criterion to come out "significantly higher" than all other weights. Let us try to quantify the case: if the weight for the extreme most important criterion has a value of less than 50%, it is in fact no longer the extreme most important criterion, as all other weights together already exceed the weigth of the extreme most important criterion.

We calculate the *weight ratio WR* of different AHP scales as ratio of the maximum weight for one "extreme most important" criterion over the sum off all other weights.

$$WR = \frac{w_{\max}}{(n-1)w_{\min}} \tag{5}$$

With eq. 3 we get

$$WR = \frac{M}{n-1} \tag{6}$$

The fundamental AHP scale covers a weight ratio of nine, or one order of magnitude. Scales in category 3 (M > 9) *expand* the weight ratio, and improve the discrimination of weights, scales of category 2 (*M* < 9) *compress* the weight ratio, and weights come closer together.

If we set the threshold at a weight ratio of one, *i.e* the most important criterion gets the same weight as the rest of criteria, we get

$$\frac{M}{n-1} = 1 \text{ or simply } n = M + 1 \tag{7}$$

The fundamental AHP 1 to 9 scale and the other scales in category 1 cross the threshold at *n* = 10 criteria. Scales of category 2 cross the threshold at three or four criteria, and scales of category 3 keep the maximum weight significantly higher, even beyond the number of ten criteria.

Interestingly, Saaty's fundamental 1 to 9 scale seems to represent a kind of compromise: for two criteria the extreme most important criterion receives a weight of 90%: most decision makers would probably accept it as a fair representation of their judgment. The threshold is reached at the number of ten criteria, which is within the recommendation, to limit the number of criteria to the *Magical Number Seven plus or Minus Two* (Saaty & Ozdemir, 2003).

Based on the weight bound and weight ratio we can now compare the scales, up to which number $n_{\max}$ of criteria they are applicable under the condition that the extreme most important criterion gets a weight of more than 50% (table 3).

Scale	AHP	Inv-lin.	Bal-n	Log	Root	Kocz	Power	Geom
$n_{\max}$	10	4	10	4	4	3	82	257

Table 3. Maximum number of criteria $n_{\max}$ for different AHP scales based on a weight threshold of 50% for a single extrem most important criterion.

It can be seen that for many practical applications of medium complexity (decision problems with more than three or four criteria) the introduced threshold based on the scale limitation *eliminates the use of the scales in category two*.

4. Weight uncertainty

Using the judgment scale *x* = 1 … 9, we will now look at small variations of judgements by Δ*x* = ±0.5 and their impact on the resulting weights Δ*x*/Δ*w*. A variation of ±0.5 corresponds to the resolution of the scale, when rounding values to an integer. With this we will get an idea about the uncertainties of the resulting weights. The calculation is done using eq. 2, determining the weight differences based on *x* = *c* ±0.5 in eq. 2, with *c* as the

scaled values from table 2 for the different scales. The result for *n* = 7 criteria is shown in figure 2.
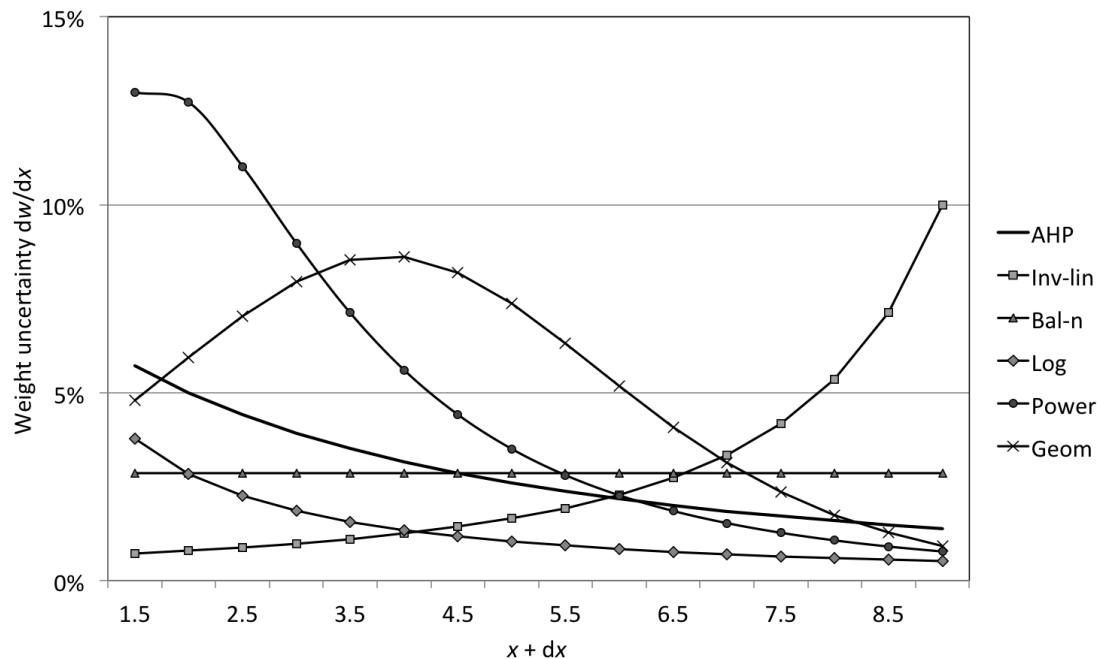

Figure 2. Weight uncertainty as a function of judgments (*x* = 1 to 9) for different AHP scales and *n* = 7 criteria.

Looking at the original *AHP 1 to 9 scale*, we see a steady decrease from 5.7% (*x* = 1, equal) to 1.4% (*x* = 9, extreme), *i.e. judgements with lower preference result in a higher uncertainty than judgements with higher preference*. The steady decrease can also be seen for the *power scale*, but the absolute percentage is higher, going from 13% (*x* = 1, *equal*) down to 0.8% (*x* = 9, *extreme*). The *inverse-linear scale* shows the opposite behaviour: low uncertainties for low preferences (0.7%/1.3% for *x* = 1, equal), increasing to 10%/6.2% for higher preferences (*x* = 9, extreme). The uncertainty for priorities of the *balanced-n scale* is with 3% constant over the whole judgement range. The *geometric scale* has its maximum at the judgement values *x* = 3, 4 (*moderate* and *moderate plus*). This maximum changes with a change in the number of criteria; for *n* = 3 criteria it lies at *x* = 1, 2 (*equal important* and *slight more important*). *Balanced scale* and *root square scale* are not shown in fig. 2. The *balanced scale* behaves similar to the *inverse-linear scale* with lower uncertainty for *x* = 9, and the *root square scale* similar to the *logarithmic scale* with lower weight uncertainty for *x* = 1.

We see from these values that it is important for all scales to identify the weight uncertainties, because they are not negligible, and in some decision problems they could impact the result.

5. Weight dispersion

As a measure of weight dispersion *WD* for different AHP scales we calculate the standard deviation of the differences of weights *w* for each transition on the 1 to 9 judgment scale,

$$WD = \sqrt{\sum(\Delta w_i - \overline{\Delta w})^2/7} \tag{8}$$

with

$$\Delta w_i = w_{x=i} - w_{x=i+1}$$

for i = 1 … 8.

The weight differences Δw_i are calculated using the scale functions from table 2 and eq. 2. Evenly or more uniform distributed priorities will give a lower standard deviation than unevenly distributed weights. Ideally, the balanced-n scale should show a standard deviation of zero.

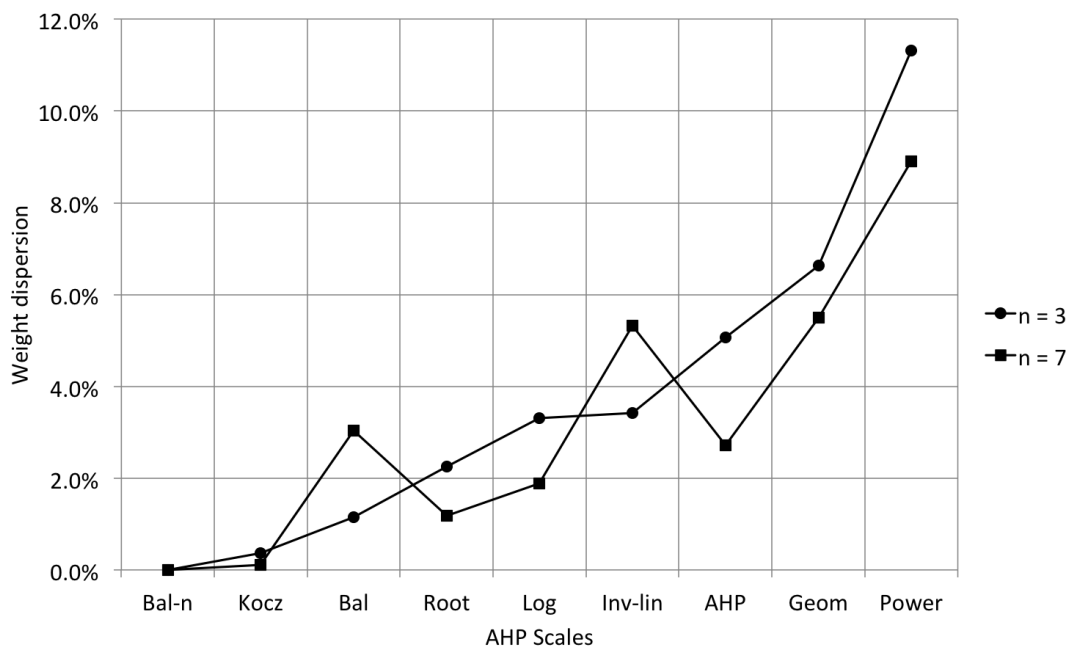Figure 3 shows the result for n = 3 and n = 7 criteria, sorted by increasing weight dispersion.

Figure 3. Weight dispersion for different AHP scales. Parameter is the number of decision criteria n.

As expected, the balanced-n scale for n = 3 and n = 7 has a standard deviation of zero. The balanced scale, as well as the inverse-linear scale of Ma and Zheng, show much lower weight dispersion for n = 3 compared to n = 7. The most unevenly distributed weights can be seen for the geometric and power scale. The AHP scale is somewhere in the middle (3% to 5%), and the scales of category 2 (reduced matrix entry values) show a more uniform weight distribution than all other scales.

6. AHP adaptive scales

All published AHP scales under study in table 2 are functions of the judgement x and are not depending on the number of criteria. We have shown, based on the balanced scale, that the number of criteria n has an impact on the result (eq. 2), because we have to consider the complete decision matrix. We now can easily design a scale, where we keep the weight of the extreme most important criterion at a constant value over the number n

of criteria, and where we will have a constant weight range of approx. 10 dB (or one order of magnitude) for all *n*. We call this an adaptive scale.

We calculate M^* to keep the maximum weight w_{max} at 90% for all possible *n*:

$$\frac{M}{n+M-1} = 0.9 = w_{max} \qquad (4)$$

$$M^* = M(n-1) \qquad (5)$$

We choose the scale function *c* as a function of *x* to have the form $c = x^y$ in order to keep it linear with the logarithm of the stimulus *x*. With $c_{max} = M^* = M(n\text{-}1)$, $c_{min} = x = 1$ and $M = 9$ we get:

$$y = 1 + \frac{\ln(n-1)}{\ln(9)} \qquad (6a)$$

and as a result the adaptive scale

$$c = x^{1+\frac{\ln(n-1)}{\ln 9}} \qquad (6b)$$

For *n* = 1 it represents the original AHP scale, for *n* = 10 it represents the power scale. For all *n* the maximum possible weight is 90%.

We can also apply this concept to the balanced-n scale (eq. 4) using $w_{max} = 0.9$ and we get the *adaptive-balanced scale* with:

$$w_{bal} = \frac{1}{n} + \frac{0.9-\frac{1}{n}}{8}(x-1) \qquad (7a)$$

or

$$c = \frac{w_{bal}(x)}{1-w_{bal}(x)}(n-1) \qquad (7b)$$

This scale function keeps the maximum weight at 90% independent from the number of criteria; at the same time the weights are equally distributed over the range [0.1, 0.9], as for the balanced-n scale.

7. Comparison of AHP scales

We can now compare and discuss all scales as shown in table 4 based on the criteria described in the previous paragraphs.

| Cat | Scale | Weight Boundary | | Weight Ratio | | n_{max} | Max. Weight Uncertainty | | Weight Dispersion |
		n=3	n=7	n=3	n=7		n=3	n=7	n = 7
1	AHP scale	82%	60%	4.5	1.5	10	9.5%	5.7%	2.7%
	Inv-lin						6.8%	10%	5.3%
	Balanced						4.0%	6.2%	3.0%
	Balanced-n						3.0%	2.9%	0.0%
2	Log Scale	62%	36%	1.7	0.5	**4**	6.5%	3.8%	1.9%
	Root square	60%	33%	1.5	0.5		4.6%	2.7%	1.2%
	Koczkodaj	50%	25%	1.0	0.3	3	1.4%	0.8%	0.1%
3	Power	99%	93%	40	14	> 10	20%	13%	8.9%
	Geometric	99.8%	98%	128	43		8.6%	8.6%	5.5%
	Adaptive	90%		9.0			13%	11.5%	7.6%
	Adapt.-bal						3.5%	4.7%	0.0%

Table 4. AHP scale comparison

For *category 1* scales weight boundary, weight range and the max. number of criteria are the same. Differences can be seen in the max. *weight uncertainty* and *weight dispersion*. By concept, the balanced-n scale has no weight dispersion, weights are equally distributed over the judgment range. The original AHP scale has a lower weight dispersion and slightly lower uncertainty than the inverse-linear scale. Based on weight uncertainty and weight dispersion the balanced-n scale is preferable compared to the original AHP scale.

The critical point for all *category 2* scales is the compression of the weight ratio. It yields to a less significant discrimination of weights, and based on the threshold for the maximum number of criteria, they should not be applied for problems with more than three or four criteria.

Category 3 scales expand the weight range and make the discrimination of priorities more significant. The geometric scale is preferable compared to the power scale, as it has a lower weight uncertainty and also a lower weight dispersion. The newly proposed adaptive-balanced scale combines a higher weight range with low uncertainty and equally distributed weights.

Comparing the scales across all categories, balance-n and adaptive-balanced scale show the best values. A further advantage is that their weight uncertainty is constant over the whole judgment range 1 to 9, and the uncertainty does not exceed 5% for up to ten criteria.

8. Implementation and practical example

The author has implemented a web based free AHP online software (AHP-OS), which can handle complete hierarchies of complex decision problems (Goepel, 2014). As the software does not store the results, but the decision makers' judgments, it is possible to

analyze results by switching between different scales. Weight uncertainty is estimated based on randomised variations of all judgments by ±0.5 on the judgment scale.

Calculation shown in this paper were made based on the specific case that one criterion is judged superior to all others, and that we have consistent matrices. Therefore we show an examples of a realistic project, to demonstrate the findings of this paper. We will also consider the consistency ratio CR in relation to the different scales.

We take the example of "Buying a house" from Saaty (1990), because all necessary input data are given in this paper. The decision matrix has eight criteria. The calculated weights for this example are shown in table 5.

No	Criterion	w_{AHP} %	Rank
1	Size of house	17.3	3
2	Transportation	5.4	5
3	Neighbhood	18.8	2
4	Age of house	1.8	8
5	Yard space	3.1	7
6	Modern facilities	3.6	6
7	Gen. condition	16.7	4
8	Financing	33.3	1

Table 5. Priorities and ranking of criteria for the example taken from Saaty (1990)

The criterion Financing (8) has the highest weight of 33%, three criteria (1, 3, 7) have a weight of approx. 18% and the remaining four criteria (2, 4, 5, 6) a weight from 2% to 5%. Figure 4 shows the change of these weights, when we apply different AHP scales.
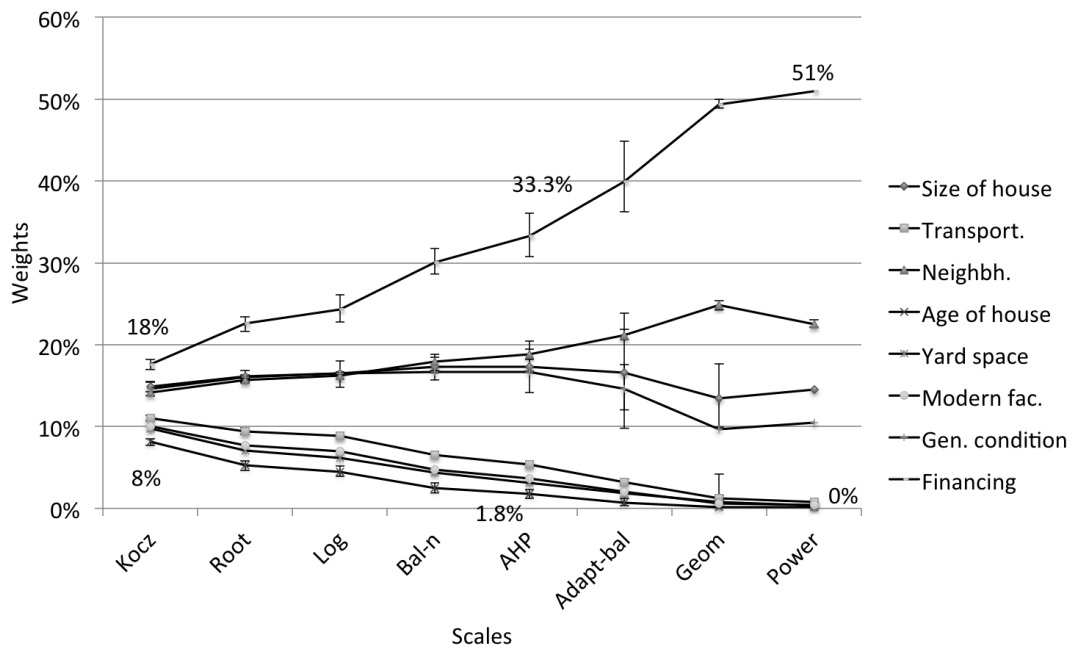

Figure 4. Changes of weights for the example with 8 criteria as a function of different AHP scales. The error bars indicate the weight uncertainty based on a randomized variation of judgment values by ±0.5 on the judgment scale.

As expected, scales of category 2 *compress* the weight range, scales of category 3 *expand* the weight range. The weight range for the Koczkodaij scale (highest compression) is 10%, *i.e.* all calculated priorities lie between 8% and 18%. For the power and geometric scale the range expands to 50%.

The weight of the criterion with the highest weight (8, Financing) changes from 18% on the Koczkodaij scale to 51% on the power scale. Weights in the mid-range change less under different scales, weights in the low range increase for category 2 scales, and decrease for category 3 scales.

The example from Saaty (1990) also shows the evaluation of three alternatives, house A, B, C. Table 6 compares the results using different scales. Weights of criteria and alternatives are both evaluated using the same scale. AHP, adaptive-balanced and balanced-n scale results are close with no change of the ranking of alternatives; root square and geometric scale show a change in the ranking.

Scale	Alternative A		Alternative B		Alternative C	
	w %	Rank	w %	Rank	w %	Rank
AHP	39.6	1	34.1	2	26.3	3
Adapt-bal	39.7	1	34.1	2	26.2	3
Bal-n	40.8	1	30.6	2	28.6	3
Root	38.4	1	30.1	**3**	31.5	**2**
Geom	40.5	2	43.6	**1**	15.9	**3**

Table 6. Alternative evaluation for example from Saaty (1990) under different scales.

Consistency Ratio CR

Although the author did not investigate the impact of the scales on the consistency ratio *CR* in detail, a few observations could be made going through a couple of actual projects. Scales of category 2 usually lower *CR*, scales of category 3 increase *CR*. For category 1 the inverse-linear and balanced-n scale lower *CR*. For the adaptive-balanced scale, the change of CR depends on the number of criteria; for up to seven or eight criteria *CR* is often the same or slightly lower compared to the original AHP scale.

9. Conclusion

The discussion about the *right scale* for the analytic hierarchy process is ongoing for many years. With this paper the author has shown that it is possible without complex mathematics or computer simulations (*e.g.* Dong et al, 2008) to derive some fundamental relations for the evaluation of different AHP scales. Based on the specific case that a decision maker prefers one single criterion above all others, we can derive a simple analytical relation to calculate the AHP weights using the row geometric mean method. For this specific case the result is the same as for the eigenvector method. Due to the limitation of the scale to a maximum judgment value (usually nine on the fundamental AHP scale), we can also calculate the maximum and minimum possible weights.

In a first step it was then shown that the balanced scale has to be corrected to take into account the number of criteria, in order to yield to equally distributed priorities across the judgment range. A modification was presented, and the corrected scale is called *balanced-n* scale.

AHP scales were categorized in three categories, depending on their maximum entry value to the decision matrix. For the final comparison of scales *weight boundaries*, *weight ratio*, *weight uncertainties* and *weight dispersion* over the judgment range were used. To overcome the limitations of the maximum weight, an adaptive-balanced scale was proposed and included in the comparison. In addition to the theoretical calculations a typical decision example was evaluated using the different scales of this study.

Based on the comparisons, the main findings can be summarized as follows.

1. Scales reducing the entry ratio into the decision matrix to lower values than nine (category 2) *compress* the calculated weights, making weight discrimination more difficult. Based on a threshold of 50% for one single most preferred criterion their application for decision problems with more than three or four criteria is not recommended.

2. Scales extending the entry ratio into the decision matrix (category 3) *expand* the calculated weights, making weight discrimination easier. At the same time they show higher weight dispersion and the weight uncertainties increase. Practical projects also show an increase of the consistency ratio CR.

3. The original AHP scale seems to present a kind of compromise with respect to the maximum number of criteria, weight dispersion and weight uncertainty. For all category 1 scales, only the (corrected) balanced-n scale improves weight dispersion and weight uncertainty in comparison to the original AHP scale. Practical projects also indicate an improvement of the consistency ratio CR for the balanced-n scale.

4. The proposed adaptive-balanced scale overcomes the problem of a change of the maximum weight depending on the number of criteria. This scale is identical with the balanced-n scale, but keeps the weight ratio at nine for any number of criteria. It results in evenly distributed weights across the judgment range, and is with respect to weight uncertainty still preferable to the original AHP scale.

10. Acknowledgments

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Annex 1: AHP weights as a function of judgments

Let *DM* be a *n* x *n* decision matrix, where the first criterion is *x*-times more important than all others. Then the first matrix element is "1", and the rest of the first matrix row is filled with (n-1)-times *x*. The first matrix column is filled with (n-1)-times 1/x.

$$DM = \begin{pmatrix} \mathbf{1} & x & x \\ 1/x & \mathbf{1} & 1 \\ 1/x & 1 & \mathbf{1} \end{pmatrix} \tag{a1}$$

To calculate the priorities, we use the Row Geometric Mean Method (RGGM), as the decision matrix is consistent and the result will be the same as for the right eigen vector.

$$\mathrm{RGGM} \rightarrow \begin{pmatrix} (x^{n-1})^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \\ \left(\frac{1}{x}\right)^{1/n} \end{pmatrix} \tag{a2}$$

The resulting weights (priorities) for the first criterion is the normalized geometric mean of the first row.

$$w_{\mathrm{AHP}} = \frac{(x^{n-1})^{\frac{1}{n}}}{(x^{n-1})^{\frac{1}{n}} + (n-1)(x^{-1})^{\frac{1}{n}}} \tag{a3}$$

With some rearrangement

$$w_{\mathrm{AHP}} = \frac{1}{1 + \frac{(n-1)(x^{-1})^{\frac{1}{n}}}{(x^{n-1})^{\frac{1}{n}}}} = \frac{1}{1 + \frac{(n-1)x^{-\frac{1}{n}}}{x \cdot x^{-\frac{1}{n}}}} = \frac{1}{1 + \frac{(n-1)}{x}} \tag{a4}$$

we get the simple relation

$$w_{\mathrm{AHP}} = \frac{x}{x+n-1} \tag{a5}$$

qed.